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Self-consistent field method from a τ-functional viewpoint*

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Abstract. A unified aspect of the self-consistent field (SCF) method and the τ-functional method is presented. SCF theory in the τ-functional space F_∞ manifestly results in a gauge theory of fermions and then a collective motion appears as a motion of fermion gauges with a common factor. This provides a new algebraic tool for the microscopic understanding of fermion many-body systems.

1. Introduction

To go beyond the perturbative method with respect to collective variables [1], we should have a very strong interest in the algebro-geometric relation between the method truncating a collective motion out of a fully parametrized self-consistent field (SCF) manifold and the τ-functional method constructing integrable equations (Hirota's equations [2]) in soliton theory [3]. The relation between τ-functions and coherent state representatives was first pointed out by D'Ariano and Rasetti [4] for an infinite-dimensional harmonic charged Fermi gas. If we stand by their observation, we may assert that the so-called SCF method [5] has presented a theoretical scheme for an integrable sub-dynamics on a certain infinite-dimensional fermion Fock space. Then we are forced to investigate the relation between the collective submanifold and various subgroup orbits in the fully parametrized SCF manifold.

The usual Hartree–Fock (HF) theory is formulated by a variational method to optimize the energy expectation value by a Slater determinant (S-det) and to obtain a variational equation for orbitals in the S-det [5]. The set of particle–hole-type pair operators of the fermions with n single-particle states is closed under Lie multiplication and forms the basis of the Lie algebra u_n [6]. The u_n Lie algebra of the pair operators generates the Thouless transformation [7] which induces a representation of the corresponding $U(n)$ group. The $U(n)$ canonical transformation transforms an S-det with m particles to another S-det. Any S-det is obtained by a $U(n)$ canonical transformation of a given reference S-det (the Thouless theorem). The Thouless transformation provides an exact generator coordinate representation of the fermion state vectors in which the generator coordinate is the $U(n)$ group and the generating wavefunction is an independent-particle one [8]. This is the generalized coherent state representation [9].

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In soliton theory on a group manifold, the transformation group to cover the solution for the soliton equation is an infinite-dimensional Lie group whose infinitesimal generator of the corresponding Lie algebra is expressed as an infinite-order differential operator of the associative affine Kac–Moody algebra. The space of the complex polynomial algebra is realized in terms of a Fock space of *infinite-dimensional fermions*. The infinite-order differential operator is represented in terms of infinite-dimensional fermions. Then the soliton equation becomes nothing other than differential equations defining group orbits of the highest-weight vector in the infinite-dimensional Fock space F_∞ [3]. The generating wavefunction, i.e. the generalized coherent state representation is just the S-det by the Thouless theorem and provides a key to elucidating the relation between the HF wavefunction and the τ-function in soliton theory [3]. Up to now, however, the relation between them has been insufficiently investigated. Both methods have been constructed on the associative Lie algebra generated by the fermion pair operators but descriptions of dynamical fermion systems by them have looked very different at first glance. In this paper, first, we present a unified aspect of the SCF method on the group $U(n)$ and the τ-functional method on the group in the soliton theory, aiming to get a close connection between the concept of a mean-field potential and the gauge of fermions inherent in the SCF method and making the role of the *loop group* [10] relating them clear. This will give a new algebraic tool for a microscopic understanding of fermion many-body systems.

In the above abstract fermion Fock spaces, we find common features in both methods, i.e. the SCF method and the τ-functional method to construct *integrable equations* for the soliton as follows.

(a) *Each solution space* is described as *Grassmannian*, i.e. a group orbit of the corresponding vacuum state.
(b) The former may implicitly explain the Plücker relation, not in terms of bilinear differential equations defining a finite-dimensional Grassmannian, but in terms of the physical concept of the quasi-particle and vacuum and the mathematical language of coset space and the coset variable. Various boson expansion methods are built on the Plücker relation to hold the Grassmannian. The latter asserts that the soliton equations are nothing but the bilinear differential equations. It gives *a boson representation of the Plücker relation*. These have been reported by one of the present authors (SN) at the *6th Int. Wigner Symp.* (1999) [11].

On the other hand, we become aware of the following two points of difference between the methods.

(a) The former is built on a *finite*-dimensional Lie algebra but the latter on an *infinite*-dimensional one.
(b) The former has an SCF *Hamiltonian* consisting of a fermion one-body operator, which is derived from a functional derivative of the expectation value of the fermion Hamiltonian by a ground-state wavefunction. In contrast, the latter introduces artificially a *fermion Hamiltonian* of a one-body-type operator as a *boson mapping operator* from states on fermion Fock space to corresponding ones on τ-functional space.

Getting over the difference due to the dimension of fermions, we ask the following. How is a *collective submanifold* which is truncated through the SCF equation related to a *subgroup orbit* in the infinite-dimensional Grassmannian by the τ-functional method? To obtain a microscopic understanding of cooperative phenomena, the concept of collective motion is introduced in relation to time-dependent (TD) variation of the self-consistent mean field. Independent-particle motion is described in terms of particles referring to a stationary mean field. The TD variation of the TD mean field is attributed to couplings between the collective and the independent-particle motions and couplings among quantum fluctuations of the TD mean field [12]. There is one-to-one correspondence between the *mean-field potential* and the vacuum state of the system. Decoupling of collective motion out of fully parametrized TDHF dynamics corresponds to a truncation of the *integrable sub-dynamics* from a fully parametrized TDHF manifold. The collective submanifold is a collection of collective paths developed by the SCF equation. Collectivity of each path reflects the *geometrical attribute of the Grassmannian* independent of the SCF Hamiltonian. Then the collective submanifold should be understood in relation to the collectivity of various subgroup orbits in the Grassmannian. The collectivity arises through interference among interacting fermions and links with the concept of a mean-field potential. The perturbative method has been considered to be useful for describing the *periodic* collective motion with large amplitude [1, 12]. If we do not break the group structure of the Grassmannian in the perturbative method, the *loop group* may work under that treatment.

Thus we note the following point in both methods: various subgroup orbits consisting of a *loop* path may *exist infinitely* in the fully parametrized TDHF manifold. They must satisfy an infinite set of Plücker relations to hold the Grassmannian. As a result, the finite-dimensional Grassmannian on the circle S^1 is identified with an infinite-dimensional one. Namely, the τ-functional method works as an algebraic tool to classify the subgroup orbits. The SCF Hamiltonian is able to exist in the infinite-dimensional Grassmannian. Then the SCF theory can be rebuilt on the infinite-dimensional fermion Fock space and hence on the τ-functional space. The infinite-dimensional fermions are introduced through a Laurent expansion of the finite-dimensional fermions with respect to degrees of freedom of fermions related to the mean-field potential. Inversely, the collectivity of the mean-field potential is attributed to gauges of interacting infinite-dimensional fermions and interference among fermions is elucidated via the Laurent parameter. These are described with the use of affine Kac–Moody algebra according to the idea of Dirac's positron theory [13]. Algebro-geometric structure of *infinite*-dimensional fermion many-body systems can be realized in the *finite*-dimensional case. Furthermore, we clarify the algebraic mechanism for truncating the collective submanifold.

In sections 2 and 3, preserving the conventional SCF method, the TDHF theory is reconstructed on an infinite-dimensional fermion Fock space F_∞. In section 4 the TDHF theory is transcribed to a τ-functional space. The TDHF theory manifestly results in a *gauge theory of fermions* inherent in the usual SCF method and the collective motion appears as a motion of fermion gauges with a common factor. The role of the soliton equation (Plücker relation) and the TDHF equation is made clear. The algebraic mechanism bringing the concept of particle and collective motions is clarified and the close connection between collective variables and the spectral parameter in soliton theory is induced. Finally, in the last section, a summary and some concluding remarks are given.

2. Conventional SCF method

We consider a finite many-fermion system with n single-particle states. Let c_α and $c_\alpha^\dagger$ ($\alpha = 1, \ldots, n$) be the annihilation–creation operators of the fermion. Owing to the anticommutation relations among them

$$\{c_\alpha, c_\beta^\dagger\} = \delta_{\alpha\beta} \qquad \{c_\alpha, c_\beta\} = \{c_\alpha^\dagger, c_\beta^\dagger\} = 0 \tag{2.1}$$

fermion pair operators span a Lie algebra. The pair operators $c_\alpha^\dagger c_\beta$ satisfy the Lie commutation relation

$$[c_\alpha^\dagger c_\beta, c_\gamma^\dagger c_\delta] = \delta_{\beta\gamma} c_\alpha^\dagger c_\delta - \delta_{\alpha\delta} c_\gamma^\dagger c_\beta. \tag{2.2}$$

The brackets $\{\cdot\,,\cdot\}$ and $[\cdot\,,\cdot]$ denote the anticommutator and the commutator, respectively. The operator $c_\alpha^\dagger c_\beta$ generates a canonical transformation $U(g)$ $(= \mathrm{e}^{\gamma_{\alpha\beta}c_\alpha^\dagger c_\beta})$ which is specified by a $U(n)$ matrix g as

$$\begin{aligned} &U(g)c_\alpha^\dagger U^{-1}(g) = c_\beta^\dagger g_{\beta\alpha} \qquad && U(g)c_\alpha U^{-1}(g) = c_\beta g_{\beta\alpha}^* \\ &U^{-1}(g) = U(g^{-1}) = U(g^\dagger) \qquad && U(gg') = U(g)U(g') \end{aligned} \tag{2.3}$$

where g is represented as below and satisfies the orthogonality condition

$$\begin{aligned} &g = \mathrm{e}^\gamma \qquad \gamma^\dagger = -\gamma \qquad (n \times n \text{ anti-Hermitian matrix}) \\ &g^\dagger g = gg^\dagger = 1_n \qquad (n\text{-dimensional unit matrix}). \end{aligned} \tag{2.4}$$

We use the dummy index convention to sum up repeated indices unless there is a possibility of misunderstanding. The symbols $\dagger$, $*$ and T denote Hermitian conjugation, complex conjugation and transposition, respectively. Let $|0\rangle$ be a free particle vacuum $c_\alpha|0\rangle = 0$ $(\alpha = 1, \ldots, n)$ and $|\phi\rangle$ be an m-particle S-det $|\phi\rangle = c_m^\dagger \cdots c_1^\dagger|0\rangle$. Under (2.2) and (2.3), $U(g)$ transforms $|\phi\rangle$ to another S-det (the Thouless transformation) [7]

$$U(g)|\phi\rangle = (c^\dagger g)_m \cdots (c^\dagger g)_1|0\rangle \overset{d}{=} |g\rangle \qquad U(g)|0\rangle = |0\rangle. \tag{2.5}$$

The m-particle S-det is an exterior product of m single-particle states. Such states are called *simple* states. The set of all simple states together with the equivalence relation identifying states different from each other only in phases with the same state, constitutes a manifold known as a Grassmannian Gr_m. The Gr_m is an orbit of the group given through equation (2.5). In the Gr_m we can make an expression called the Plücker coordinate which has played an important role in the algebraic construction of soliton theory in its early stages [14],

$$\begin{aligned} &U(g)|\phi\rangle = \sum_{n\geqslant\alpha_m>\cdots>\alpha_1\geqslant 1} v_{\alpha_m,\ldots,\alpha_1}^{m,\ldots,1} c_{\alpha_m}^\dagger \cdots c_{\alpha_1}^\dagger|0\rangle \\ &v_{\alpha_m,\ldots,\alpha_1}^{m,\ldots,1} = \det\begin{bmatrix} g_{\alpha_1,1} & \cdots & g_{\alpha_1,m} \\ \vdots & & \vdots \\ g_{\alpha_m,1} & \cdots & g_{\alpha_m,m} \end{bmatrix} \qquad \text{(Plücker coordinate)}. \end{aligned} \tag{2.6}$$

Being induced from calculations of a determinant, we easily find that the Plücker coordinate has a relation

$$\sum_{i=1}^{m+1}(-1)^{i-1} v_{\beta_i,\alpha_{m-1},\ldots,\alpha_1}^{m,\ldots,1} v_{\beta_{m+1},\ldots,\beta_{i+1},\beta_{i-1},\ldots,\beta_1}^{m,\ldots,1} = 0 \qquad \text{(Plücker relation)} \tag{2.7}$$

where the indices denote the distinct sets $1 \leqslant \alpha_1, \ldots, \alpha_{m-1} \leqslant n$ and $1 \leqslant \beta_1, \ldots, \beta_{m+1} \leqslant n$. The Plücker relation is equivalent to a bilinear identity equation

$$\sum_{\alpha=1}^{n} c_\alpha^\dagger U(g)|\phi\rangle \otimes c_\alpha U(g)|\phi\rangle = \sum_{\alpha=1}^{n} U(g)c_\alpha^\dagger|\phi\rangle \otimes U(g)c_\alpha|\phi\rangle = 0. \tag{2.8}$$

The bilinear equation has a more general form

$$\sum_{\alpha=1}^{n} c_\alpha^\dagger U(g)|\phi_k\rangle \otimes c_\alpha U(g)|\phi_l\rangle = \sum_{\alpha=1}^{n} U(g)c_\alpha^\dagger|\phi_k\rangle \otimes U(g)c_\alpha|\phi_l\rangle = 0 \qquad (n \geqslant k \geqslant l \geqslant 0) \tag{2.9}$$

where $|\phi_k\rangle$ and $|\phi_l\rangle$ denote a k-particle simple state and an l-particle one, respectively. It is noted that the Gr_m is essentially an $SU(n)$ group manifold since the phase equivalence theorem does hold.

According to Rowe *et al* [15], we start with a geometrical aspect of the SCF method, i.e. the TDHF equation, in the following way: let us consider the time-dependent Schrödinger equation $\mathrm{i}\hbar\partial_t\Psi = H\Psi$ with a Hamiltonian

$$H = h_{\beta\alpha}c_\beta^\dagger c_\alpha + \tfrac{1}{2}\langle\gamma\alpha|\delta\beta\rangle c_\gamma^\dagger c_\delta^\dagger c_\beta c_\alpha \tag{2.10}$$

where $h_{\beta\alpha}$ and $\langle\gamma\alpha|\delta\beta\rangle$ denote a single-particle Hamiltonian and a matrix element of an interaction potential, respectively. This equation is linear but generally shows dispersive behaviour. A TDHF equation gives a dynamics constrained to a nonlinear space on the Gr_m. Suppression of the dispersion of a wavefunction results in the nonlinear TDHF equation in which a non-dispersive solution is a path on the Gr_m. The starting point for the TDHF theory lies in an extremal condition of an action integral

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\,\mathcal{L}(g(t)) = 0 \qquad \mathcal{L}(g(t)) \stackrel{d}{=} \langle\phi|U(g^\dagger(t))(\mathrm{i}\hbar\partial_t - H)U(g(t))|\phi\rangle. \tag{2.11}$$

To obtain an explicit expression for the TDHF equation, we calculate an expectation value of one- and two-body operators for the S-det (2.5). Using the canonical transformation (2.3), we have

$$W_{\alpha\beta} \stackrel{d}{=} \langle\phi|U(g^\dagger)c_\beta^\dagger c_\alpha U(g)|\phi\rangle = (g^\dagger)_{\beta'\beta}(g^{\mathrm{T}})_{\alpha'\alpha}\langle\phi|c_{\beta'}^\dagger c_{\alpha'}|\phi\rangle = \sum_{\alpha'=1}^{m} g_{\alpha\alpha'}g_{\alpha'\beta}^\dagger \tag{2.12}$$

$$\begin{aligned}\langle\phi|U(g^\dagger)c_\gamma^\dagger c_\delta^\dagger c_\beta c_\alpha U(g)|\phi\rangle &= (g^\dagger)_{\gamma'\gamma}(g^\dagger)_{\delta'\delta}(g^{\mathrm{T}})_{\beta'\beta}(g^{\mathrm{T}})_{\alpha'\alpha}\langle\phi|c_{\gamma'}^\dagger c_{\delta'}^\dagger c_{\beta'}c_{\alpha'}|\phi\rangle\\ &= W_{\alpha\gamma}W_{\beta\delta} - W_{\alpha\delta}W_{\beta\gamma}.\end{aligned} \tag{2.13}$$

From equations (2.12) and (2.13), we obtain an energy functional, expectation value of the Hamiltonian (2.10)

$$\begin{aligned}&H[W] \stackrel{d}{=} \langle\phi|U(g^\dagger)HU(g)|\phi\rangle = h_{\beta\alpha}W_{\alpha\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta]W_{\alpha\gamma}W_{\beta\delta}\\ &[\gamma\alpha|\delta\beta] = \langle\gamma\alpha|\delta\beta\rangle - \langle\gamma\beta|\delta\alpha\rangle.\end{aligned} \tag{2.14}$$

We also obtain the HF Hamiltonian $H_{\mathrm{HF}}[W]$ by projecting the original Hamiltonian onto the Gr_m,

$$H_{\mathrm{HF}}[W] = \mathcal{F}_{\alpha\beta}[W]c_\alpha^\dagger c_\beta \qquad \mathcal{F}_{\alpha\beta} = \frac{\delta H[W]}{\delta W_{\beta\alpha}} = h_{\alpha\beta} + [\alpha\beta|\gamma\delta]W_{\delta\gamma}. \tag{2.15}$$

The Lagrange function $\mathcal{L}(g(t))$ in (2.11) is computed as

$$\mathcal{L}(g(t)) = \tfrac{1}{2}\mathrm{i}\hbar(g_{ab}^\dagger\dot{g}_{ba} + g_{ai}^\dagger\dot{g}_{ia} - \dot{g}_{ab}^\dagger g_{ba} - \dot{g}_{ai}^\dagger g_{ia}) - H[W] \tag{2.16}$$

using $\partial_t U(g^\dagger(t))U(g(t)) + U(g^\dagger(t))\cdot\partial_t U(g(t)) = 0$. The condition (2.11) gives the TDHF equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{g}^\dagger}\right) - \frac{\partial\mathcal{L}}{\partial g^\dagger} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{g}}\right) - \frac{\partial\mathcal{L}}{\partial g} = 0 \tag{2.17}$$

and then we obtain a compact form of the TDHF equation $\mathrm{i}\hbar\partial_t g(t) = \mathcal{F}[W\{g(t)\}]g(t)$. The time evolution of the S-det (2.5) is given by $\mathrm{i}\hbar\partial_t U(g(t))|\phi\rangle = H_{\mathrm{HF}}[W(g(t))]U(g(t))|\phi\rangle$.

3. SCF method in F_∞

We will construct the SCF method, i.e. the TDHF theory on infinite-dimensional fermion Fock space F_∞, according to the essence of the physical picture in the conventional SCF method. Although it is different from the usual space-coordinate construction of infinite-dimensional fermions [16], we will start from the following: the canonical transformation (2.3) preserves a unitary equivalence for the original single-particle Schrödinger equation, that is to say, it induces a kind of iso-spectral deformation. We assume that the Schrödinger equation with a TD potential holds an iso-spectrum under time evolution of the potential. Taking not only the *Pauli principle* but also *time–energy indeterminacy* into account, we can rewrite the anticommutation relations (2.1) as

$$\{c_\alpha(t), c^\dagger_\beta(t')\} = \delta_{\alpha\beta}\delta(t-t') \qquad \{c_\alpha(t), c_\beta(t')\} = \{c^\dagger_\alpha(t), c^\dagger_\beta(t')\} = 0. \tag{3.1}$$

If the TD potential has periodicity in time T, the eigenfunction has the same periodicity. Through Laurent expansion of the fermion operators, infinite-dimensional fermion operators including both particle spectra and Laurent spectra can be obtained as

$$\begin{aligned} c_\alpha(t) &= \sum_{r\in\mathbb{Z}} \left(\frac{\hbar}{T}\right)^{1/2} \psi^*_{nr+\alpha} z^r \qquad c^\dagger_\alpha(t) = \sum_{r\in\mathbb{Z}} \left(\frac{\hbar}{T}\right)^{1/2} \psi_{nr+\alpha} z^{-r} \\ \delta(t-t') &= \frac{\hbar}{T} \sum_{r\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}\hbar\frac{2\pi}{T} r(t-t')} \end{aligned} \tag{3.2}$$

where $\mathbb{Z}$ means a set of integers and indices α and r are called the labels on particle spectra and on Laurent spectra, respectively. Substitution of (3.2) into (3.1) leads to the anticommutation relations

$$\{\psi^*_{nr+\alpha}, \psi_{ns+\beta}\} = \delta_{\alpha\beta}\delta_{rs} \qquad \{\psi^*_{nr+\alpha}, \psi^*_{ns+\beta}\} = \{\psi_{nr+\alpha}, \psi_{ns+\beta}\} = 0. \tag{3.3}$$

If the canonical transformation (2.3) is time dependent and generates the time evolution of the potential, it is possible to embed the $U(n)$ group induced from (2.3) into a group induced from the canonical transformation of the infinite-dimensional fermion operators (3.3). Then, the method of describing the collective motion as motion of the TD mean-field potential in the SCF theory may be speculated to have a close connection with the soliton theory on the infinite-dimensional Fock space [3].

Suppose that the collective motion is dependent only on collective variables η and η^* with a period of time T, $\eta(t) = \eta(t+T)$ and $\eta^*(t) = \eta^*(t+T)$. The matrix γ ($\in u_n$) in the $U(n)$ matrix g (2.4) also has the same periodicity of time T via the collective variables as $\gamma(t+T) = \gamma(\eta(t+T), \eta^*(t+T)) = \gamma(\eta(t), \eta^*(t)) = \gamma(t)$. As for the ordinary perturbative method with respect to η and η^* with periodicity [1], we will represent the collective variables η and η^* as $\eta = \sqrt{\Omega}\,\mathrm{e}^{\mathrm{i}\phi}$ and $\eta^* = \sqrt{\Omega}\,\mathrm{e}^{-\mathrm{i}\phi}$ with amplitude Ω. Then we can always express the matrix γ as $\gamma(\eta, \eta^*) = \sum_{r,s\in\mathbb{Z}} \bar{\gamma}_{r,s}\eta^{*r}\eta^s = \sum_{r\in\mathbb{Z}} \gamma_r z^r$ on the Lie algebra u_n if we put $z = \mathrm{e}^{\mathrm{i}\phi}$. Regarding this expression as a relation of the Lie algebra of maps from the unit circle S^1 to the Lie algebra u_n [16], we make a Laurent expansion of the γ as

$$\gamma(z) = \sum_{r\in\mathbb{Z}} \gamma_r z^r \qquad z = \mathrm{e}^{-\mathrm{i}\hbar\omega_c t} \quad \left(\omega_c = \frac{2\pi}{T}\right) \tag{3.4}$$

where r runs in this time over a finite subset of $\mathbb{Z}$. For the anti-Hermitian condition for the $\gamma(z)$, we impose the constraints $\gamma^\dagger(z) = -\gamma(z) \mapsto \gamma^\dagger_r = -\gamma_{-r}$ and $z^{-1} = z^*$ ($|z| = 1$). We can consider these maps as *loop groups* [10].

According to Kac and Raina [17], we can introduce an infinite-dimensional Fock space F_∞ and an associative affine Kac–Moody algebra restricting ourselves to u_n algebra. Using the fundamental idea of Dirac's positron theory [13], the perfect vacuum $|\mathrm{Vac}\rangle$ and reference vacuum $|m\rangle$ are shown as

$$\begin{aligned} &\psi_{nr+\alpha}|\mathrm{Vac}\rangle = 0 \qquad \langle\mathrm{Vac}|\psi^*_{nr+\alpha} = 0 \qquad (r \leqslant -1) \\ &\psi^*_{nr+\alpha}|\mathrm{Vac}\rangle = 0 \qquad \langle\mathrm{Vac}|\psi_{nr+\alpha} = 0 \qquad (r \geqslant 0) \\ &|m\rangle = \psi_m \cdots \psi_1|\mathrm{Vac}\rangle \end{aligned} \tag{3.5}$$

with normalization conditions $\langle\mathrm{Vac}|\mathrm{Vac}\rangle = 1$ and $\langle m|m\rangle = 1$. Then we embed the free vacuum $|0\rangle$ and simple state $|\phi\rangle$ into the infinite-dimensional Fock space F_∞ as

$$|0\rangle \mapsto |\mathrm{Vac}\rangle \qquad |\phi\rangle \mapsto |m\rangle \qquad (m = 1, \ldots, n). \tag{3.6}$$

We assume a state with the Laurent spectrum corresponding to the $|0\rangle$ to be a stable state with minimal energy. This assumption means we make a choice of gauge under which we adopt the correspondence $|0\rangle \mapsto |\mathrm{Vac}\rangle$.

Using the *correspondence* between basic elements: $c^\dagger_\alpha c_\beta z^r \mapsto \tau(e_{\alpha\beta}(r)) \stackrel{d}{=} \sum_{s\in\mathbb{Z}} \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta}$ and the *normal-ordered product*: $:\psi_{nr+\alpha}\psi^\star_{ns+\beta}: \stackrel{d}{=} \psi_{nr+\alpha}\psi^\star_{ns+\beta} - \delta_{\alpha\beta}\delta_{rs}$ $(s < 0)$, let us define the following $\widehat{su_n}$ $(\subset \widehat{sl_n})$ Lie algebra:

$$\begin{aligned} &X_\gamma = \overline{X}_\gamma + \mathbb{C}\cdot c \qquad \mathbb{C}^* = -\mathbb{C} \qquad \text{(pure imaginary)} \\ &\overline{X}_\gamma = \sum_{r=-N}^{N}\sum_{s\in\mathbb{Z}} (\gamma_r)_{\alpha\beta} :\psi_{n(s-r)+\alpha}\psi^*_{ns+\beta}: \qquad \gamma_r^\dagger = -\gamma_{-r} \qquad \mathrm{Tr}\,\gamma_r = 0 \\ &[X_\gamma, c] = 0 \qquad [X_\gamma, X_{\gamma'}] = \overline{X}_{[\gamma,\gamma']} + \alpha(\gamma,\gamma')\cdot c \qquad c|m\rangle = 1\cdot|m\rangle \end{aligned} \tag{3.7}$$

$$\gamma = \begin{array}{c} \uparrow \\ h \\ \downarrow \\ \uparrow \\ p \\ \downarrow \end{array} \overset{\longleftarrow \; h \; \longrightarrow \; | \; \longleftarrow \; p \; \longrightarrow}{\left[\begin{array}{ccccc|ccccc} & \ddots & & & & & & \ddots & & \\ \gamma_{-2} & \gamma_{-1} & \gamma_0 & \gamma_1 & & \gamma_2 & & & \ddots & \\ & \gamma_{-2} & \gamma_{-1} & \gamma_0 & & \gamma_1 & \gamma_2 & & & \ddots \\ \hline \ddots & & \gamma_{-2} & \gamma_{-1} & & \gamma_0 & \gamma_1 & \gamma_2 & & \\ & \ddots & & \gamma_{-2} & & \gamma_{-1} & \gamma_0 & \gamma_1 & \gamma_2 & \\ & \ddots & & & & & & \ddots & & \end{array}\right]}$$

$$\alpha(\gamma,\gamma') = \sum_{r=-N}^{N} r\,\mathrm{Tr}(\gamma_r\gamma'_{-r}) = -\tfrac{1}{2}\,\mathrm{Tr}\left[\begin{array}{ccc|ccc} \ddots & & & & & \\ & -I_n & & & & \\ & & -I_n & & & \\ \hline & & & I_n & & \\ & & & & I_n & \\ & & & & & \ddots \end{array}\right][\overline{\gamma},\overline{\gamma}'] \tag{3.8}$$

where c and I_n denote the centre and the n-dimensional unit matrix. The matrix γ is divided into four blocks by specifying apparently occupied states h and unoccupied states p for the perfect vacuum $|\mathrm{Vac}\rangle$. Corresponding to this division, the matrix in the 2-cocycle α is also divided into four blocks in analogy with Dirac's positron theory [13] as seen in the above. $\overline{\gamma}$ and $\overline{\gamma}'$ represent off-diagonal parts of the matrices γ and γ' as

$$\overline{\gamma} \stackrel{d}{=} \left[\begin{array}{cc|cc} & & \gamma_2 & \ddots \\ & & \gamma_1 & \gamma_2 \\ \hline \gamma_{-2} & \gamma_{-1} & & \\ \ddots & \gamma_{-2} & & \end{array}\right] \qquad \overline{\gamma'} \stackrel{d}{=} \left[\begin{array}{cc|cc} & & \gamma_2' & \ddots \\ & & \gamma_1' & \gamma_2' \\ \hline \gamma_{-2}' & \gamma_{-1}' & & \\ \ddots & \gamma_{-2}' & & \end{array}\right]. \tag{3.9}$$

Note the relation $\alpha^*(\gamma, \gamma') = -\alpha(\gamma, \gamma')$ and properties $\gamma^\dagger = -\gamma$ and $\gamma'^\dagger = -\gamma'$. Now, using (3.3), (3.7) and the identity $[AB, C] = A\{B, C\} - \{A, C\}B$, adjoint actions of X_γ for ψ and ψ^* are computed as

$$\begin{aligned} [X_\gamma, \psi_{nr+\alpha}] &= \sum_{s=-N}^{N} \psi_{n(r-s)+\beta}(\gamma_s)_{\beta\alpha} \\ [X_\gamma, \psi^*_{nr+\alpha}] &= \sum_{s=-N}^{N} \psi^*_{n(r-s)+\beta}(\gamma^*_s)_{\alpha\beta}. \end{aligned} \tag{3.10}$$

Here s runs over a finite set of the integer $\mathbb{Z}$ $(= -N, -N+1, \ldots, N)$. Furthermore, using (3.10) and the operator identity called the Baker–Campbell–Hausdorff formula

$$\mathrm{e}^{X_\gamma} A \mathrm{e}^{-X_\gamma} = A + [X_\gamma, A] + \frac{1}{2!}[X_\gamma, [X_\gamma, A]] + \cdots \tag{3.11}$$

the infinite-dimensional fermion operator is transformed by the canonical transformation $U(\hat{g})(\hat{g} = \mathrm{e}^{\gamma})$, which satisfies $U^{-1}(\hat{g}) = U(\hat{g}^{-1}) = U(\hat{g}^\dagger)$ and $U(\hat{g}\hat{g}') = U(\hat{g})U(\hat{g}')$ with $\hat{g}^\dagger\hat{g} = \hat{g}\hat{g}^\dagger = 1_\infty$, into the forms

$$\begin{aligned} \psi_{nr+\alpha}(\hat{g}) &\stackrel{d}{=} U(\hat{g})\psi_{nr+\alpha}U^{-1}(\hat{g}) = \sum_{s\in\mathbb{Z}} \psi_{n(r-s)+\beta}(g_s)_{\beta\alpha} \\ \psi^*_{nr+\alpha}(\hat{g}) &\stackrel{d}{=} U(\hat{g})\psi^*_{nr+\alpha}U^{-1}(\hat{g}) = \sum_{s\in\mathbb{Z}} \psi^*_{n(r-s)+\beta}(g^*_s)_{\beta\alpha} \end{aligned} \tag{3.12}$$

where 1_∞ is an infinite-dimensional unit matrix and $\hat{g}_{nr+\alpha,ns+\beta} = (g_{s-r})_{\alpha\beta}$, $\hat{g}^\dagger_{nr+\alpha,ns+\beta} = (g^\dagger_{r-s})_{\alpha\beta}$ and

$$\begin{aligned} \delta_{rs}\delta_{\alpha\beta} &= (\hat{g}\hat{g}^\dagger)_{nr+\alpha,ns+\beta} = \sum_{t\in\mathbb{Z}} (g_t g^\dagger_{t+(r-s)})_{\alpha\beta} \\ \delta_{rs}\delta_{\alpha\beta} &= (\hat{g}^\dagger\hat{g})_{nr+\alpha,ns+\beta} = \sum_{t\in\mathbb{Z}} (g^\dagger_t g_{t-(r-s)})_{\alpha\beta}. \end{aligned} \tag{3.13}$$

Note that $\hat{g}$ forms a periodic sequence with period n and s and t run over a formally infinite set of $\mathbb{Z}$.

Let us construct the TDHF theory in F_∞. The bilinear equations (2.8) and (2.9) can be embedded into those in F_∞. Using the corresponding arguments $|\phi_k\rangle \mapsto |k\rangle$, $U(g) \mapsto U(\hat{g})$ $(= \mathrm{e}^{X_\gamma})$ and

$$\sum_{\alpha=1}^{n} c^\dagger_\alpha \otimes c_\alpha \mapsto \sum_{\alpha=1}^{n}\sum_{r\in\mathbb{Z}} c^\dagger_\alpha z^{-r} \otimes c_\alpha z^r \simeq \sum_{\alpha=1}^{n}\sum_{r\in\mathbb{Z}} \psi_{nr+\alpha} \otimes \psi^*_{nr+\alpha} \tag{3.14}$$

they are embedded into the bilinear equations on F_∞ as

$$\sum_{\alpha=1}^{n}\sum_{r\in\mathbb{Z}}\psi_{nr+\alpha}U(\hat{g})|m\rangle\otimes\psi^*_{nr+\alpha}U(\hat{g})|m\rangle$$
$$=\sum_{\alpha=1}^{n}\sum_{r\in\mathbb{Z}}U(\hat{g})\psi_{nr+\alpha}|m\rangle\otimes U(\hat{g})\psi^*_{nr+\alpha}|m\rangle=0 \qquad (m=1,\ldots,n) \tag{3.15}$$

$$\sum_{\alpha=1}^{n}\sum_{r\in\mathbb{Z}}\psi_{nr+\alpha}U(\hat{g})|k\rangle\otimes\psi^*_{nr+\alpha}U(\hat{g})|l\rangle$$
$$=\sum_{\alpha=1}^{n}\sum_{r\in\mathbb{Z}}U(\hat{g})\psi_{nr+\alpha}|k\rangle\otimes U(\hat{g})\psi^*_{nr+\alpha}|l\rangle=0 \qquad (k\geqslant l;\, k,l=1,\ldots,n). \tag{3.16}$$

Thus we arrive at the following picture: the algebra of extracting subgroup orbits made of the *loop* path from Gr_m belongs to an sl_n-reduction of gl_∞ in soliton theory. Relieving from restrictions of su_n and (3.15) and taking $\gamma\in sl_n$ with m and $k\geqslant l$ $(\in\mathbb{Z})$, equations (3.15) and (3.16) can be regarded as the bilinear equations of the reduced KP (Kadomtsev–Petviashvili) hierarchy and the modified KP in soliton theory [3]. This picture suggests the possibility of constructing the SCF method for an equation of collective motion governed by equations (3.15) and (3.16) on a bigger space sl_n than su_n. However, we must note that in the SCF method the bilinear equations (3.15) and (3.16) are considered to play the role of conditions ensuring the existence of subgroup orbits on the Gr_m different from soliton theory in which boson expressions for them become an infinite set of dynamical equations. Note also that the concept of a quasi-particle and vacuum in the SCF method on S^1 is connected to the Plücker relation by using the basic idea of Dirac's positron theory [13].

Now we attempt to embed the original two-body Hamiltonian (2.10) into the F_∞. By replacing the annihilation–creation operators of the fermions as $c_\alpha\mapsto\sum_{r\in\mathbb{Z}}\psi^*_{nr+\alpha}$ and $c^\dagger_\alpha\mapsto\sum_{r\in\mathbb{Z}}\psi_{nr+\alpha}$, we obtain

$$H_{F_\infty}=h_{\beta\alpha}\sum_{r,s\in\mathbb{Z}}\psi_{nr+\beta}\psi^*_{ns+\alpha}+\tfrac{1}{2}\langle\gamma\alpha|\delta\beta\rangle\sum_{k,l\in\mathbb{Z}r,s\in\mathbb{Z}}\psi_{nk+\gamma}\psi_{nl+\delta}\psi^*_{ns+\beta}\psi^*_{nr+\alpha}. \tag{3.17}$$

Introducing a new Laurent spectral number K, a pair operator is rewritten into a form

$$\sum_{r,s\in\mathbb{Z}}\psi_{nr+\beta}\psi^*_{ns+\alpha}\mapsto\sum_{K\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\psi_{n(s-K)+\beta}\psi^*_{ns+\alpha}. \tag{3.18}$$

To embed the SCF Hamiltonian (2.15), we introduce a general Hamiltonian on the F_∞ as

$$H_{F_\infty}=\sum_{r,s\in\mathbb{Z}}h_{ns+\beta,nr+\alpha}\psi_{ns+\beta}\psi^*_{nr+\alpha}$$
$$+\tfrac{1}{2}\sum_{r,s\in\mathbb{Z}k,l\in\mathbb{Z}}\langle nk+\gamma,nr+\alpha|nl+\delta,ns+\beta\rangle\psi_{nk+\gamma}\psi_{nl+\delta}\psi^*_{ns+\beta}\psi^*_{nr+\alpha} \tag{3.19}$$

which is equivalent to (3.17) if $h_{ns+\beta,nr+\alpha}=h_{\beta\alpha}$ and $\langle nk+\gamma,nr+\alpha|nl+\delta,ns+\beta\rangle=\langle\gamma\alpha|\delta\beta\rangle$ (equivalence conditions for H_{F_∞}) hold. To calculate the formal expectation value of (3.19) for the vector $U(\hat{g})|m\rangle$, first we do it for one- and two-body operators. Using equation (3.12) we obtain

$$\langle m|\psi_{ns+\beta}\psi^*_{nr+\alpha}|m\rangle=\delta_{sr}\delta_{\beta\alpha} \qquad (\text{for } r=0,\alpha=1,\ldots,m \text{ and for } r<0,\alpha=1,\ldots,n)$$
$$\langle m|\psi_{nk+\gamma}\psi_{nl+\delta}\psi^*_{ns+\beta}\psi^*_{nr+\alpha}|m\rangle=\delta_{kr}\delta_{\gamma\alpha}\cdot\delta_{ls}\delta_{\delta\beta}-\delta_{ks}\delta_{\gamma\beta}\cdot\delta_{lr}\delta_{\delta\alpha} \tag{3.20}$$
$$(\text{for } r(s)=0,\alpha(\beta)=1,\ldots,m \text{ and for } r(s)<0,\alpha(\beta)=1,\ldots,n).$$

Then for one- and two-body-type operators we obtain formally

$$\begin{aligned} W^f_{nr+\alpha,ns+\beta} &= \langle m|U(\hat{g}^\dagger)\psi_{ns+\beta}\psi^*_{nr+\alpha}U(\hat{g})|m\rangle \\ &= \sum_{\gamma=1}^{m} \hat{g}_{nr+\alpha,\gamma}\hat{g}^\dagger_{\gamma,ns+\beta} + \sum_{t<0}\sum_{\gamma=1}^{n} \hat{g}_{nr+\alpha,nt+\gamma}\hat{g}^\dagger_{nt+\gamma,ns+\beta} \\ &= \sum_{\gamma=1}^{m}(g_{-r})_{\alpha\gamma}(g^\dagger_{-s})_{\gamma\beta} + \sum_{t<0}\sum_{\gamma=1}^{n}(g_{t-r})_{\alpha\gamma}(g^\dagger_{t-s})_{\gamma\beta} \end{aligned} \tag{3.21}$$

$$\langle m|U(\hat{g}^\dagger)\psi_{nk+\gamma}\psi_{nl+\delta}\psi^*_{ns+\beta}\psi^*_{nr+\alpha}U(\hat{g})|m\rangle = W^f_{nr+\alpha,nk+\gamma}W^f_{ns+\beta,nl+\delta} - W^f_{nr+\alpha,nl+\delta}W^f_{ns+\beta,nk+\gamma}. \tag{3.22}$$

Thus we obtain the formal expectation value of (3.19) as

$$\begin{aligned} \langle H_{F_\infty}\rangle[W^f] &= \sum_{r,s\in\mathbb{Z}} h_{ns+\beta,nr+\alpha}W^f_{nr+\alpha,ns+\beta} \\ &\quad +\tfrac{1}{2}\sum_{r,s\in\mathbb{Z}}[nk+\gamma, nr+\alpha|nl+\delta, ns+\beta]W^f_{nr+\alpha,nk+\gamma}W^f_{ns+\beta,nl+\delta} \end{aligned} \tag{3.23}$$

$$[nk+\gamma, nr+\alpha|nl+\delta, ns+\beta] = \langle nk+\gamma, nr+\alpha|nk+\delta, ns+\beta\rangle - \langle\delta \longleftrightarrow \gamma\rangle.$$

For the Hamiltonian (3.17), from (3.23) and the equivalence conditions for H_{F_∞}, we obtain

$$\begin{aligned} \langle H_{F_\infty}\rangle[W^f] &= h_{\beta\alpha}\sum_{r,s\in\mathbb{Z}} W^f_{nr+\alpha,ns+\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta]\sum_{r,s\in\mathbb{Z}\ k,l\in\mathbb{Z}} W^f_{nr+\alpha,nk+\gamma}W^f_{ns+\beta,nl+\delta} \\ &= \sum_{k\in\mathbb{Z}}(h_k)_{\beta\alpha}\sum_{s\in\mathbb{Z}} W^f_{ns+\alpha,n(s-k)+\beta} \\ &\quad + \tfrac{1}{2}\sum_{k,l\in\mathbb{Z}}[(k,\gamma),\alpha|(l,\delta),\beta]\sum_{r\in\mathbb{Z}} W^f_{nr+\alpha,n(r-k)+\gamma}\sum_{s\in\mathbb{Z}} W^f_{ns+\beta,n(s-l)+\delta} \end{aligned} \tag{3.24}$$

$$(h_k)_{\beta\alpha} \equiv h_{\beta\alpha} \qquad [(k,\gamma),\alpha|(l,\delta),\beta] \equiv [\gamma\alpha|\delta\beta].$$

To avoid the *anomaly* in the expectation value, taking a summation over the infinite numbers, we change the one-body operator (3.21) into its normal-ordered product form as

$$\begin{aligned} (W_k)_{\alpha\beta} &\stackrel{d}{=} \langle m|U(\hat{g}^\dagger) : \tau(e_{\beta\alpha}(-k)) : U(\hat{g})|m\rangle \\ &= \sum_{r\in\mathbb{Z}}\langle m|U(\hat{g}^\dagger) : \psi_{n(r+k)+\beta}\psi^*_{nr+\alpha} : U(\hat{g})|m\rangle \\ &= \sum_{r\in\mathbb{Z}} W^f_{nr+\alpha,n(r+k)+\beta} - \sum_{r<0}\delta_{k,0}\delta_{\beta\alpha} = \sum_{r\in\mathbb{Z}}\sum_{\gamma=1}^{m}(g_{-r})_{\alpha\gamma}(g^\dagger_{-r-k})_{\gamma\beta} \end{aligned} \tag{3.25}$$

where we have used the correspondence relation between basic elements.

W_k is identical to a coefficient of the Laurent expansion of the density matrix (2.12)

$$W_{\alpha\beta}(z) = \sum_{k\in\mathbb{Z}}(W_k)_{\alpha\beta}z^k = \sum_{k\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sum_{\gamma=1}^{m}(g_s)_{\alpha\gamma}(g^\dagger_{s-k})_{\gamma\beta}z^k. \tag{3.26}$$

Changing W^f in (3.24) into its normal-ordered product and using (3.25), we obtain

$$\begin{aligned} \langle H_{F_\infty}\rangle[W] &= h_{\beta\alpha}\sum_{k\in\mathbb{Z}}(W_{-k})_{\alpha\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta]\sum_{k,l\in\mathbb{Z}}(W_{-k})_{\alpha\gamma}(W_{-l})_{\beta\delta} \\ &= \sum_{k\in\mathbb{Z}}\{h_{\beta\alpha}(W_{-k})_{\alpha\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta]\sum_{l\in\mathbb{Z}}(W_{-k+l})_{\alpha\gamma}(W_{-l})_{\beta\delta}\}. \end{aligned} \tag{3.27}$$

The result coincides with the formal Laurent polynomials of (2.14) in the sense of

$$H[W(z)] = h_{\beta\alpha} \sum_{k\in\mathbb{Z}} (W_k)_{\alpha\beta} z^k + \tfrac{1}{2}[\gamma\alpha|\delta\beta] \sum_{k,l\in\mathbb{Z}} (W_k)_{\alpha\gamma} z^k (W_l)_{\beta\delta} z^l$$
$$= \sum_{l\in\mathbb{Z}} \{h_{\beta\alpha}(W_l)_{\alpha\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta] \sum_{k\in\mathbb{Z}} (W_{l-k})_{\alpha\gamma}(W_k)_{\beta\delta}\} z^l. \tag{3.28}$$

The time dependence of the energy functional is brought through $z(t)$. To preserve the time independence, in (3.28) we put the Laurent spectrum l zero. That is to say, we may select a sub-functional as

$$\langle H_{F_\infty}\rangle[W] = h_{\beta\alpha}(W_0)_{\alpha\beta} + \tfrac{1}{2}[\gamma\alpha|\delta\beta] \sum_{k\in\mathbb{Z}} (W_k)_{\alpha\gamma}(W_{-k})_{\beta\delta} \tag{3.29}$$

which means that the Laurent spectra k and l in the first line of equation (3.27) cancel each other. That is extraction of the sub-Hamiltonian $H^{\mathrm{sub}}_{F_\infty}$ out of equation (3.17) as

$$H^{\mathrm{sub}}_{F_\infty} = h_{\beta\alpha} \sum_{s\in\mathbb{Z}} \psi_{ns+\beta}\psi^*_{ns+\alpha} + \tfrac{1}{2}[\gamma\alpha|\delta\beta] \sum_{k\in\mathbb{Z}} \sum_{r,s\in\mathbb{Z}} \psi_{n(r-k)+\gamma}\psi_{n(s+k)+\delta}\psi^*_{ns+\beta}\psi^*_{nr+\alpha}. \tag{3.30}$$

The above extraction permits us to interpret $H_{F_\infty}[W]$ as

$$H[W(z)]|_{z^0} = \frac{1}{2\pi \mathrm{i}} \oint \frac{H[W(z)]}{z}\, \mathrm{d}z.$$

Therefore, we adopt here equation (3.29) as the energy functional for the u_n algebra on F_∞. Through the variation

$$\delta\langle H_{F_\infty}\rangle[W] = \sum_{k\in\mathbb{Z}} (\mathcal{F}_{-k})_{\alpha\beta}\delta(W_k)_{\beta\alpha} \qquad (\mathcal{F}_k)_{\alpha\beta} \overset{d}{=} h_{\alpha\beta}\delta_{k,o} + [\alpha\beta|\gamma\delta](W_k)_{\delta\gamma} \tag{3.31}$$

we obtain an SCF Hamiltonian on F_∞ very similar to the formal Laurent expansion of H_{HF} (2.15) on Gr_m as

$$H_{F_\infty;\mathrm{HF}} = \sum_{k\in\mathbb{Z}} \sum_{s\in\mathbb{Z}} (\mathcal{F}_k)_{\alpha\beta} : \psi_{n(s-k)+\alpha}\psi^*_{ns+\beta} : . \tag{3.32}$$

For the TDHF equation on F_∞, the state vector $U(\hat{g})|m\rangle$ is required to satisfy the variational principle

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t\, L(\hat{g}) = 0 \qquad L(\hat{g}) = \langle m|U(\hat{g}^\dagger)(\mathrm{i}\partial_t - H_{F_\infty})U(\hat{g})|m\rangle \tag{3.33}$$

where we use $\hbar = 1$ here and hereafter. First, by using $U(\hat{g}) = \mathrm{e}^{X_\gamma}$ we obtain the following relations:

$$\delta_{\hat{g}} \int \mathrm{d}t\, \langle m|U(\hat{g}^\dagger)\mathrm{i}\partial_t U(\hat{g})|m\rangle = \delta_{\hat{g}} \int \mathrm{d}t\, \langle m|\mathrm{i}\partial_t|m\rangle$$
$$+\delta_{\hat{g}} \int \mathrm{d}t\, \langle m|\mathrm{i}\partial_t X_\gamma - \frac{1}{2!}[X_\gamma, \mathrm{i}\partial_t X_\gamma] + \cdots + |m\rangle \tag{3.34}$$

$$\mathrm{i}\partial_t X_\gamma = \sum_{r=-N}^{N} \sum_{s\in\mathbb{Z}} \{(\mathrm{i}\partial_t\gamma_r)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : + (\gamma_r)_{\alpha\beta}\mathrm{i}\partial_t : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} :\} + \mathrm{i}\partial_t\mathbb{C} \tag{3.35}$$

where we have used (3.11) and (3.12). From the definition of $\tau(e_{\alpha\beta}(r))$ and the normal-ordered product, we can calculate the time differentiation of the second term in the curly brackets of equation (3.35) as

$$\mathrm{i}\partial_t \sum_{s\in\mathbb{Z}} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} := \mathrm{i}\partial_t : \tau(e_{\alpha\beta}(r)) := \mathrm{i}r\partial_t \ln z : \tau(e_{\alpha\beta}(r)) : . \tag{3.36}$$

Assuming that the parameter of the Laurent expansion is given by $z = \mathrm{e}^{-\mathrm{i}\omega_c t}$, equation (3.35) is rewritten as

$$\mathrm{i}\partial_t X_\gamma = \sum_{r=-N}^{N} \sum_{s\in\mathbb{Z}} (D_{r;t}(\gamma_r)_{\alpha\beta}) : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : +\mathrm{i}\partial_t \mathbb{C} \qquad D_{r;t} \stackrel{d}{=} \mathrm{i}\partial_t + r\omega_c. \tag{3.37}$$

It is seen that in the first term of the right-hand side of (3.34) a time evolution of the reference vacuum through $z(t)$ has no influence on the variation with respect to $\hat{g}$. Concerning the time evolution of X_γ (3.37) a time differential ∂_t acting on $\gamma_r(t)$ and on ψ and ψ^* through $z(t)$ is transformed into a covariant differential $D_{r;t}$ with a connection $r\omega_c$ which acts only on the $\gamma_r(t)$ from the gauge-theoretic viewpoint. We denote the covariant differential simply as D. Therefore, we can put $\langle m|\mathrm{i}\partial_t|m\rangle = 0$ and $\mathbb{C} = 0$ since it has no influence on the energy functional (3.29). Then the time-differential term in (3.33) is calculated as

$$\begin{aligned} U(\hat{g}^\dagger)\mathrm{i}\partial_t U(\hat{g}) &= \mathrm{i}\partial_t X_\gamma + \frac{1}{2!}[\mathrm{i}\partial_t X_\gamma, X_\gamma] + \frac{1}{3!}[[\mathrm{i}\partial_t X_\gamma, X_\gamma], X_\gamma] + \cdots \\ &= \overline{X}_{D\gamma} + \sum_{k\geqslant 2} \frac{1}{k!}[\cdots[\mathrm{i}\partial_t X_\gamma, X_\gamma], \ldots], X_\gamma] + \cdots. \end{aligned} \tag{3.38}$$

Using equation (3.7) and the symbol D for the covariant differential, each commutator is calculated as

$$\begin{aligned} [\mathrm{i}\partial_t X_\gamma, X_\gamma] &= \overline{X}_{[D\gamma,\gamma]} - \tfrac{1}{2}\,\mathrm{Tr}\begin{bmatrix} -I & \\ & I \end{bmatrix} [\overline{D\gamma}, \overline{\gamma}] \\ [[\mathrm{i}\partial_t X_\gamma, X_\gamma], X_\gamma] &= \overline{X}_{[[D\gamma,\gamma],\gamma]} - \tfrac{1}{2}\,\mathrm{Tr}\begin{bmatrix} -I & \\ & I \end{bmatrix} [[\overline{D\gamma, \gamma}], \overline{\gamma}] \\ &\ \ \vdots \\ [\cdots[\mathrm{i}\partial_t X_\gamma, X_\gamma], \ldots], X_\gamma] &= \overline{X}_{[\cdots[D\gamma,\gamma],\ldots],\gamma]} - \tfrac{1}{2}\,\mathrm{Tr}\begin{bmatrix} -I & \\ & I \end{bmatrix} [\overline{\cdots[D\gamma, \gamma], \ldots}], \overline{\gamma}]. \end{aligned} \tag{3.39}$$

In the above $\overline{M}$ denotes off-diagonal parts of any matrix M and the infinite-dimensional unit matrix I_∞ is abbreviated simply as I. Substituting (3.39) into (3.38) and using $D_r \stackrel{d}{=} D_{r;t}$ and $\hat{g} = \mathrm{e}^\gamma$, we obtain

$$U(\hat{g}^\dagger)\mathrm{i}\partial_t U(\hat{g}) = \overline{X}_{\hat{g}^\dagger D\hat{g}} + \mathbb{C}(\hat{g}^\dagger D\hat{g})$$

$$\hat{g}^\dagger D\hat{g} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & \ddots \\ g_1^\dagger & g_0^\dagger & g_{-1}^\dagger & & \\ & g_1^\dagger & g_0^\dagger & g_{-1}^\dagger & \\ & & g_1^\dagger & g_0^\dagger & g_{-1}^\dagger \\ \ddots & & & & \ddots \end{bmatrix} \begin{bmatrix} \ddots & & & & \ddots \\ D_{-1}g_{-1} & D_0 g_0 & D_1 g_1 & & \\ & D_{-1}g_{-1} & D_0 g_0 & D_1 g_1 & \\ & & D_{-1}g_{-1} & D_0 g_0 & D_1 g_1 \\ \ddots & & & & \ddots \end{bmatrix}$$

$$\mathbb{C}(\hat{g}^\dagger D\hat{g}) = -\tfrac{1}{2}\,\mathrm{Tr}\begin{bmatrix} -I & \\ & I \end{bmatrix} \sum_{k\geqslant 2} \frac{1}{k!}[\overline{\cdots[D\gamma, \gamma], \cdots}], \overline{\gamma}]. \tag{3.40}$$

The expectation value for the reference vacuum is expressed as

$$\langle m|U(\hat{g}^\dagger)\mathrm{i}\partial_t U(\hat{g})|m\rangle = \sum_{s\in\mathbb{Z}} \sum_{\alpha=1}^{m} \sum_{\gamma=1}^{n} (g_s^\dagger)_{\alpha\gamma}(D_{s;t}g_s)_{\gamma\alpha} + \mathbb{C}(\hat{g}^\dagger D\hat{g}). \tag{3.41}$$

Using $\mathbb{C}(\hat{g}^\dagger D\hat{g}) - \mathbb{C}(D\hat{g}^\dagger \cdot \hat{g}) = 0$ which is proved later, we obtain an explicit expression for the $L(\hat{g})$ as

$$L(\hat{g}) = \tfrac{1}{2}\sum_{s\in\mathbb{Z}}\sum_{\alpha=1}^{m}\sum_{\gamma=1}^{n}\{(g_s^\dagger)_{\alpha\gamma}(D_{s;t}g_s)_{\gamma\alpha} - (D_{-s;t}g_s^\dagger)_{\alpha\gamma}(g_s)_{\gamma\alpha}\} - \langle H_{F_\infty}\rangle[W]. \tag{3.42}$$

Thus $L(\hat{g})$ is nothing other than the coefficient of z^0 in the Laurent expansion of $L(g(z))$ (2.16).

We give the TDHF equation for $\hat{g}$ identical with the Laurent expansion of $\mathrm{i}\partial_t g(t) = \mathcal{F}[W\{g(t)\}]g(t)$ and $\mathrm{i}\partial_t U(g(t))|\phi\rangle = H_{\mathrm{HF}}[W(g(t))]U(g(t))|\phi\rangle$. Demand on the extremal condition in (3.32) leads to

$$D_t\hat{g} = \mathcal{F}(\hat{g})\hat{g} \qquad \mathcal{F}(\hat{g}) \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & & \\ & & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 & \\ & & & \mathcal{F}_{-1} & \mathcal{F}_0 & \mathcal{F}_1 \\ \ddots & & & & & \ddots \end{bmatrix}. \tag{3.43}$$

Defining matrix elements $(\mathcal{F}_r^c)_{\alpha\beta}(\hat{g},\omega_c) \stackrel{d}{=} \omega_c\sum_{s\in\mathbb{Z}} s(g_s g_{s-r}^\dagger)_{\alpha\beta}$, equation (3.43) is transformed to

$$\begin{aligned} &\mathrm{i}\partial_t\hat{g} = \mathcal{F}^p(\hat{g})\hat{g} \qquad \mathcal{F}^p(\hat{g}) \stackrel{d}{=} \mathcal{F}(\hat{g}) - \mathcal{F}^c(\hat{g}) \\ &(\mathcal{F}_r^p)_{\alpha\beta} \stackrel{d}{=} (\mathcal{F}_r - \mathcal{F}_r^c)_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma} - \omega_c\sum_{s\in\mathbb{Z}} s(g_s g_{s-r}^\dagger)_{\alpha\beta} \end{aligned} \tag{3.44}$$

introducing $\widehat{D}_t \stackrel{d}{=} \mathrm{i}\partial_t + H^c_{F_\infty;\mathrm{HF}}$, this time which is cast into that on the state vector $U(\hat{g})|m\rangle$ as

$$\begin{aligned} &\widehat{D}_t U(\hat{g})|m\rangle = H_{F_\infty;\mathrm{HF}}U(\hat{g})|m\rangle \qquad H^c_{F_\infty;\mathrm{HF}} \stackrel{d}{=} \sum_{r,s\in\mathbb{Z}}(\mathcal{F}_r^c)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : \\ &\mathrm{i}\partial_t U(\hat{g})|m\rangle = H^p_{F_\infty;\mathrm{HF}}U(\hat{g})|m\rangle \qquad H^p_{F_\infty;\mathrm{HF}} \stackrel{d}{=} \sum_{r,s\in\mathbb{Z}}(\mathcal{F}_r^p)_{\alpha\beta} : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : \end{aligned} \tag{3.45}$$

which suggest symmetry breaking and the occurrence of collective motion due to recovery of symmetry. Suppose that $\hat{g}$ to diagonalize $\mathcal{F}^p$ in $H^p_{F_\infty;\mathrm{HF}}$ and $U(\hat{g})|m\rangle$ to do $\mathcal{F}^c$ in $H^c_{F_\infty;\mathrm{HF}}$ are determined spontaneously when $\hat{g} \simeq \hat{g}^0\mathrm{e}^{-\mathrm{i}\hat{\epsilon}t}$ and $\partial_t\hat{g}^0 = 0$. Using the definition of $\mathcal{F}^c$ we have $\omega_c\Gamma(\hat{g}^0) = \mathcal{F}(\hat{g}^0)\hat{g}^0 - \hat{g}^0\hat{\epsilon}$ where

$$\Gamma(\hat{g}^0) \stackrel{d}{=} \begin{bmatrix} \ddots & & & & & \ddots \\ & -g^0_{-1} & 0 & g^0_1 & & \\ & & -g^0_{-1} & 0 & g^0_1 & \\ & & & -g^0_{-1} & 0 & g^0_1 \\ \ddots & & & & & \ddots \end{bmatrix} \qquad \hat{\epsilon} \stackrel{d}{=} \begin{bmatrix} \ddots & & & & \\ & \epsilon & & & \\ & & \epsilon & & \\ & & & \epsilon & \\ & & & & \ddots \end{bmatrix} \tag{3.46}$$

and $g_r z^r \propto \mathrm{e}^{-\mathrm{i}(\epsilon+\omega_c r I_n)t}$. Thus the quasi-particle energy ϵ ($\epsilon_{\alpha\beta} = \epsilon_\alpha\delta_{\alpha\beta}$) and the boson energy ω_c are unified into a gauge phase. The HF theory on Gr_m has not the obviously collective term (3.46) and leads inevitably to $\omega_c\Gamma(\hat{g}^0) = 0$. $\hat{g}^0$ must be composed of only a block-diagonal $g_0^0 = \exp\gamma_0$ where γ_0 ($\in su_n$) is a block-diagonal matrix of γ (3.8).

Equation (3.45) gives the time evolution of *particle degrees of freedom*. Thus we obtain a common language, an *infinite-dimensional Grassmannian and an affine Kac–Moody algebra*, to discuss the relation between SCF and soliton theories. The SCF theory under level one on F_∞ is nothing other than the zeroth order of the Laurent expansion on Gr_m. Through construction of the SCF theory on F_∞, the explicit algebraic structure of the SCF theory on F_∞ is made clear as it is just the gauge theory inherent in the SCF theory. *Mean-field potential degrees of freedom* occur from the *gauge degrees of freedom* of fermions and the fermions make pairs among them absorbing change of gauges. The sub-Hamiltonian (3.30) exhibits such a phenomenon in the u_n algebra, which allows us to interpret the absorption of the gauge as a coherent property of fermion pairs. Thus the SCF theory can be regarded as a method to determine self-consistently or *spontaneously* both the quasi-particle energy $\epsilon_\alpha(\hat{g})$ and the boson energy ω_c which is due to the time evolution of the *fermion gauge*. Then it becomes possible to say that *both the energies have been unified into the gauge phase*.

Let ϵ and ϵ^* be parameters specifying a continuous deformation of *loop* path on Gr_m. Using the notation in (3.7) and calculating in a similar way to (3.40), $\mathrm{e}^{-X_\gamma}\partial_\epsilon \mathrm{e}^{X_\gamma}$ is obtained as

$$\begin{aligned} \mathrm{e}^{-X_\gamma}\partial_\epsilon \mathrm{e}^{X_\gamma} &= \overline{X}_{\hat{g}^{-1}\partial_\epsilon\hat{g}} + \mathbb{C}(\hat{g}^{-1}\partial_\epsilon\hat{g}) \\ \hat{g}^{-1}\partial_\epsilon\hat{g} &= \partial_\epsilon + \partial_\epsilon(\mathbb{C}\cdot 1) + \partial_\epsilon\gamma + \sum_{k\geqslant 2}\frac{1}{k!}[\cdots[\partial_\epsilon\gamma,\gamma],\ldots],\gamma]. \end{aligned} \tag{3.47}$$

Due to $\mathrm{Tr}\,\gamma_r = 0$ $\overline{X}_\gamma$ reads $\overline{X}_\gamma = \sum_{\gamma=-N}^{N}\sum_{s\in\mathbb{Z}}(\gamma_r)_{\alpha\beta} : \psi_{n(s-\gamma)+\alpha}\psi^*_{ns+\beta}:$ and $\mathrm{e}^{-X_\gamma}\partial_\epsilon \mathrm{e}^{X_\gamma}$ is computed to be

$$\mathrm{e}^{-X_\gamma}\partial_\epsilon \mathrm{e}^{X_\gamma} = \sum_{r,s\in\mathbb{Z}}(\hat{g}^{-1}\partial_\epsilon\hat{g})_r : \psi_{n(s-r)+\alpha}\psi^*_{ns+\beta} : + \sum_{s<0}\mathrm{Tr}(\hat{g}^{-1}\partial_\epsilon\hat{g})_0. \tag{3.48}$$

From equations (3.47) and (3.48), we obtain $\mathbb{C}(\hat{g}^{-1}\partial_\epsilon\hat{g}) = \sum_{s<0}\mathrm{Tr}(\hat{g}^{-1}\partial_\epsilon\hat{g})_0 = 0, (\hat{g}^{-1}\partial_\epsilon\hat{g})_0 \in sl_n$. We also obtain $\mathbb{C}(\hat{g}^\dagger\partial_{\epsilon^*}\hat{g}) = \mathbb{C}(\partial_{\epsilon^*}\hat{g}^\dagger\cdot\hat{g}) = 0$ and $\mathbb{C}(\hat{g}^\dagger D\hat{g}) = \mathbb{C}(D\hat{g}^\dagger\cdot\hat{g}) = 0$. We define infinitesimal generators of the collective submanifold as follows:

$$\begin{aligned} X_{\theta^\dagger} &\stackrel{d}{=} \mathrm{i}\partial_\epsilon U(\hat{g})\cdot U(\hat{g})^\dagger = \overline{X}_{\theta^\dagger} + \mathbb{C}(\mathrm{i}\partial_\epsilon\hat{g}\cdot\hat{g}^\dagger) \qquad & \theta^\dagger &\stackrel{d}{=} \mathrm{i}\partial_\epsilon\hat{g}\cdot\hat{g}^\dagger \\ X_\theta &\stackrel{d}{=} \mathrm{i}\partial_{\epsilon^*}U(\hat{g})\cdot U(\hat{g})^\dagger = \overline{X}_\theta + \mathbb{C}(\mathrm{i}\partial_{\epsilon^*}\hat{g}\cdot\hat{g}^\dagger) \qquad & \theta &\stackrel{d}{=} \mathrm{i}\partial_{\epsilon^*}\hat{g}\cdot\hat{g}^\dagger \end{aligned} \tag{3.49}$$

where terms $\mathbb{C}(\cdots)$ vanish. The infinitesimal generators as functions of ϵ and ϵ^* are changed to differential operators. Using this idea, we developed a theory for large-amplitude collective motions [18]. From $\partial_{\epsilon^*}\langle\hat{g}|\partial_\epsilon|\hat{g}\rangle - \partial_\epsilon\langle\hat{g}|\partial_{\epsilon^*}|\hat{g}\rangle$ and equations (3.7) and (3.25), we obtain the *weak* orthogonality condition [1, 12]

$$1 = \langle\hat{g}|[X_\theta, X_{\theta^\dagger}]|\hat{g}\rangle = \sum_{\alpha=1}^{m}\sum_{\gamma=1}^{n}\sum_{r\in\mathbb{Z}}([\theta,\theta^\dagger]_r)_{\alpha\gamma}(W_{-r})_{\gamma\alpha} - \tfrac{1}{2}\mathrm{Tr}\begin{bmatrix} -I & \\ & I \end{bmatrix}[\bar{\theta},\bar{\theta}^\dagger]. \tag{3.50}$$

We can treat the equation of collective motion and the collective submanifold along the idea of Lax pairs [19] for the construction of integrable systems. Any path on the collective submanifold can be represented as $\hat{g}(t) = \hat{g}^0(\epsilon(t),\epsilon^*(t))\,\mathrm{e}^{-\mathrm{i}\hat{\epsilon}(\epsilon(t),\epsilon^*(t))t}$, in which the collective part corresponding to a *loop* path on the Gr_m is given by $\hat{g}|_c = \hat{g}^0(\epsilon,\epsilon^*)\,\mathrm{e}^{-\mathrm{i}\hat{\epsilon}(\epsilon,\epsilon^*)t}$ and $\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \frac{\mathrm{d}\epsilon^*}{\mathrm{d}t} = 0$. For the $\hat{g}|_c$, equations (3.43) and (3.49) are converted, respectively, to curvature equations which should vanish to satisfy integrability conditions for ϵ, ϵ^* and t. Although for simplicity we took $\mathbb{C}(\hat{g}^{-1}\partial_\epsilon\hat{g})$, etc to vanish from the condition of sl_n, even if they have non-zero constant values with respect to $\hat{g}$, the above remarks are still maintained.

4. SCF method in τ-functional space

Along the soliton theory in the infinite-dimensional fermion Fock space [3, 17, 20, 21], we transcribe the TDHF theory in F_∞ to that in τ-functional space. A Heisenberg subalgebra $\mathcal{S}$ [17] is given by

$$\mathcal{S} = \bigoplus_{k\neq 0} \Lambda_k + \mathbb{C}\cdot c, \Lambda_k \stackrel{d}{=} \sum_{i\in\mathbb{Z}} :\psi_i\psi^*_{i+k}: \qquad (k\in\mathbb{Z}) \tag{4.1}$$

from which the boson algebra is obtained as $[\Lambda_k, \Lambda_l] = k\delta_{k+l,0}$. Λ_k is called the shift operator and Λ_0 belongs to the centre. We take only $c = 1$ (level-one case). Then the boson mapping operator is introduced as $\sigma_m \stackrel{d}{=} \langle m|e^{H(x)}$ where $H(x) = \sum_{j\geqslant 1} x_j\Lambda_j$ is the *Hamiltonian in the τ-functional method* [3]. By which the following isomorphism is described as σ_m; $F^{(m)} \mapsto B^{(m)} = \mathbb{C}(x_1, x_2, \ldots)$ and $|m\rangle \mapsto 1$, then

$$\Lambda_k \mapsto \frac{\partial}{\partial x_k} \qquad \Lambda_{-k}\mapsto kx_k \qquad (k>0) \qquad \Lambda_0 \mapsto m \tag{4.2}$$

where $F^{(m)}$ and $B^{(m)}$ denote m-charged fermion space and the corresponding boson space, respectively, and the degree is defined by $\deg(x_j) = j$. The contravariant Hermitian form on the $B^{(m)}$ is given as

$$\langle 1|1\rangle = 1 \qquad \left(\frac{\partial}{\partial x_k}\right)^\dagger = kx_k \qquad \langle P|Q\rangle = P^*\left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \ldots\right)Q(x)|_{x=0} \tag{4.3}$$

where the P^* means the complex conjugation of all the coefficients of polynomials P and $x = (x_1, x_2, \ldots)$.

Then the group orbit of the highest-weight vector $|m\rangle$ under the action $U(\hat{g})$ $(= e^{X_a})$ of $GL(\infty)$ is mapped to a space of the τ-function $\tau_m(x,\hat{g}) = \langle m|e^{H(x)}U(\hat{g})|m\rangle$.

We construct the representation in $B^{(m)}$ in the reduction to the $\widehat{sl_n}$. Let the generating series be

$$\Psi(p) = \sum_{j\in\mathbb{Z}} p^j\psi_j \qquad \Psi^*(p) = \sum_{j\in\mathbb{Z}} p^{-j}\psi^*_j \qquad (p\in\mathbb{C}\backslash 0). \tag{4.4}$$

The algebra X_a $(= \sum_{i,j\in\mathbb{Z}} a_{ij} : \psi_i\psi^*_j : +\mathbb{C}\cdot 1) \in \widehat{sl_n}$ must satisfy the following conditions:

$$\begin{aligned} &\text{(i)} \qquad a_{i+n,j+n} = a_{ij} \qquad (i,j\in\mathbb{Z}) \\ &\text{(ii)} \qquad \sum_{i=1}^{n} a_{i,i+jn} = 0 \qquad (j\in\mathbb{Z}). \end{aligned} \tag{4.5}$$

From (i) and $\Lambda_{jn} = \sum_{i\in\mathbb{Z}} : \psi_i\psi^*_{i+jn} :$ $(j\in\mathbb{Z})$, $[X_a, \Lambda_{jn}] = 0$. This tells us that $\tau_m(x,\hat{g})(\hat{g}\in\widehat{sl_n})$ is independent of x_{jn}. Note that Λ_{jn} does not satisfy (ii). Using (i), the generating function is rewritten as

$$\Psi(p)\Psi^*(q) = \sum_{i,j\in\mathbb{Z}} \psi_i\psi^*_j p^i q^{-j} = \sum_{i,j\in\mathbb{Z}} \psi_{i+n}\psi^*_{j+n} p^i q^{-j} p^n q^{-n} \tag{4.6}$$

where we must set the condition $p^n = q^n$, i.e. $q = \epsilon^s p, \epsilon = e^{2\pi i/n}$ $(s = 0, 1, \ldots, n-1)$. Then the *vertex representation* of : $\Psi(p)\Psi^*(q)$: becomes

$$\begin{aligned} &:\mathbf{\Psi}(p)\mathbf{\Psi}^*(\epsilon^* p) := \frac{1}{1-\epsilon^s}\{\epsilon^{-ms}\Gamma(p,\epsilon^s p) - 1\} \\ &\Gamma(p,\epsilon^s p) = \left\{\exp\sum_{j\geqslant 1}(1-\epsilon^{sj})p^j x_j\right\}\left\{\exp\sum_{j\geqslant 1}\frac{-(1-\epsilon^{-sj})}{j}p^{-j}\frac{\partial}{\partial x_j}\right\}. \end{aligned} \tag{4.7}$$

Introducing the Schur polynomials by the generating function, $\exp \sum_{k\geqslant 1}^{\infty} x_k p^k = \sum_{k\geqslant 0} S_k(x) p^k$ [3, 20], the explicit expression for any element is obtained as

$$\sigma_m;\, X_a = \sum_{i,j\in\mathbb{Z}} a_{ij} : \psi_i \psi_j^* : +\mathbb{C}\cdot 1 \quad \mapsto \quad \sum_{i,j} a_{ij} \widetilde{z}_{ij}(x, \widetilde{\partial}_x) + \mathbb{C}\cdot 1$$
$$\widetilde{z}_{ij}(x, \widetilde{\partial}_x) = \sum_{\mu,\nu\geqslant 0, k\geqslant 0} S_{i+k+\mu-m}(x) S_{-j-k+\nu+m}(-x) S_\mu(-\widetilde{\partial}_x) S_\nu(\widetilde{\partial}_x) - \delta_{ij}\cdot 1 \qquad (j \leqslant 0) \tag{4.8}$$

which is independent on all x_{jn} and $\widetilde{\partial}_x \stackrel{d}{=} (\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2}, \ldots)$.

With the use of the above descriptions for the τ-functional method, we can transcribe easily the fundamental equations (3.15) and (3.45) for the TDHF theory on $U(\hat{g})|m\rangle \subset F^{(m)}$ into the corresponding τ-function $\subset B^{(m)}$ in the following forms.

(a) $U(\hat{g})|m\rangle (U(\hat{g}) = \mathrm{e}^{X_\gamma};\, X_\gamma \in \widehat{su_n} \subset \widehat{sl_n}) \mapsto$ the τ-function;

$$\tau_m(x, \hat{g}) = \langle m|\mathrm{e}^{H(x)} U(\hat{g})|m\rangle \qquad \frac{\partial}{\partial x_{jn}} \tau_m(x, \hat{g}) = 0. \tag{4.9}$$

(b) The quasi-particle and vacuum states $\mapsto$ Hirota's bilinear equation (see [2, 3]);

$$\sum_{\alpha=1}^{n} \sum_{r\in\mathbb{Z}} \psi_{nr+\alpha} U(\hat{g})|m\rangle \otimes \psi_{nr+\alpha}^* U(\hat{g})|m\rangle = 0$$
$$\mapsto \sum_{j\geqslant 0} S_j(-2y) S_{j+1}(\tilde{D}) \exp\left(\sum_{s\geqslant 1} y_s D_s\right) \tau_m(x, \hat{g}) \tau_m(x, \hat{g}) = 0. \tag{4.10}$$

(c) The TDHF equation on $U(\hat{g})|m\rangle \mapsto$ the TDHF equation on $\tau_m(x, \hat{g})$;

$$\mathrm{i}\partial_t U(\hat{g}(t))|m\rangle = H^p_{F_\infty}(\hat{g}(t)) U(\hat{g}(t))|m\rangle$$
$$\mapsto \mathrm{i}\partial_t \tau_m(x, \hat{g}(t)) = H^p_{F_\infty}(x, \widetilde{\partial}_x, \hat{g}(t)) \tau_m(x, \hat{g}(t)) \tag{4.11}$$

in which using equation (4.8), $H^p_{F_\infty}(x, \widetilde{\partial}_x, \hat{g})$ is given as

$$H^p_{F_\infty}(x, \widetilde{\partial}_x, \hat{g}) = \sum_{r,s\in\mathbb{Z}} (\mathcal{F}^p_r(\hat{g}))_{\alpha\beta} \widetilde{z}_{n(s-r)+\alpha, ns+\beta}(x, \widetilde{\partial}_x)$$
$$(\mathcal{F}^p_r(\hat{g}))_{\alpha\beta} = h_{\alpha\beta}\delta_{r,0} + [\alpha\beta|\gamma\delta](W_r)_{\delta\gamma} - \omega_c \sum_{s\in\mathbb{Z}} s(g_s g^\dagger_{s-r})_{\alpha\beta} \tag{4.12}$$
$$(W_r)_{\alpha\beta} = \sum_{\gamma=1}^{m} \sum_{s\in\mathbb{Z}} (g_s)_{\alpha\gamma} (g^\dagger_{s-r})_{\gamma\beta}.$$

$D = (D_1, D_2, \ldots)$ denotes Hirota's bilinear differential operator [2] and $\widetilde{D} = (D_1, \frac{1}{2}D_2, \ldots)$.

Algebraic manipulation of the extraction of the subgroup orbits on the Gr_m is just the method of the integrable equation (4.10) for the su_n ($\subset sl_n$) reduced KP hierarchy.

The trajectories of the TDHF equation are running in their various subgroup orbits. If we only have to know the form of X_γ deciding the subgroup orbits by the soliton equations, we can construct a Hamiltonian made of only the elements of X_γ. Then the Hamiltonian becomes integrable on the subgroup orbit.

We are now in a position to consider the collective submanifold. We will clarify more clearly the relation of the concept of particle and collective motions in the TDHF theory to that of the soliton theory from the *loop group viewpoint*. The identification by Pressely and Segal [10] surprisingly connects the Hilbert space $\mathcal{H}^{(n)} = \mathcal{L}^2(S^1; \mathbb{C}^n)$ with the standard Hilbert space $\mathcal{H} = \mathcal{H}^{(1)} = \mathcal{L}^2(S^1; \mathbb{C})$ by an obvious lexicographic correspondence between their orthonormal basis. The $\mathcal{L}^2(S^1; \mathbb{C}^n)$ denotes square summable $\mathbb{C}^n$-valued functions on the circle. Then we construct a one-component fermion operator

$$\tilde{c}(u) = \sum_{\alpha=1}^{n} u^{\alpha-1} c_\alpha(u^n) \qquad \tilde{c}^\dagger(u) = \sum_{\alpha=1}^{n} u^{-(\alpha-1)} c_\alpha^\dagger(u^n). \tag{4.13}$$

Conversely, for given $\tilde{c} \in \mathcal{H}$, we obtain $(c_\alpha) \in \mathcal{H}^{(n)}$ by

$$c_{\alpha'+1}(z) = \frac{1}{n}\sum_u u^{-\alpha'}\tilde{c}(u) \qquad c_{\alpha'+1}^\dagger(z) = \frac{1}{n}\sum_u u^{\alpha'}\tilde{c}^\dagger(u) \qquad (\alpha' = 0, \ldots, n-1) \tag{4.14}$$

u runs through the nth roots of z so as to satisfy $u = \mathrm{e}^{\mathrm{i}\theta} = \epsilon^s \mathrm{e}^{\mathrm{i}\phi/n}$ with $\epsilon = \mathrm{e}^{\mathrm{i}2\pi/n}$ $(s = 0, \ldots, n-1)$ putting $z = u^n = \mathrm{e}^{\mathrm{i}\phi}$. Using (3.2), we obtain the one-component fermion operator $\tilde{c}(u)$ and $\tilde{c}^\dagger(u)$

$$\begin{aligned}
\tilde{c}(u) &\stackrel{d}{=} \sum_{r\in\mathbb{Z}}\sum_{\alpha=1}^{n} \psi_{nr+\alpha}^* u^{nr+\alpha-1} \stackrel{d}{=} \sum_{r\in\mathbb{Z}}\sum_{\alpha=1}^{n} \tilde{\psi}_{nr+\alpha-1}^* u^{nr+\alpha-1} \\
\tilde{c}^\dagger(u) &\stackrel{d}{=} \sum_{r\in\mathbb{Z}}\sum_{\alpha=1}^{n} \psi_{nr+\alpha} u^{-(nr+\alpha-1)} \stackrel{d}{=} \sum_{r\in\mathbb{Z}}\sum_{\alpha=1}^{n} \tilde{\psi}_{nr+\alpha-1} u^{-(nr+\alpha-1)}.
\end{aligned} \tag{4.15}$$

Using equation (4.15), the shift operator Λ_k can be transcribed to that on $\mathbb{C}[u, u^{-1}]$ as

$$\begin{aligned}
:\tilde{c}^\dagger(u)\tilde{c}(u): &= \sum_{r,s\in\mathbb{Z}}\sum_{\alpha,\beta=1}^{n} :\tilde{\psi}_{n(s-r)+\alpha-1}\tilde{\psi}_{ns+\beta-1}^*: u^{nr+\alpha-1} \\
&= \sum_{r\in\mathbb{Z}}\sum_{\alpha=1}^{n}\left(\sum_{s\in\mathbb{Z}}\sum_{\beta=1}^{n} :\psi_{ns+\beta-(nr+\alpha-1)}\psi_{ns+\beta}^*:\right)u^{nr+\alpha-1} \\
&= \sum_{r\in\mathbb{Z}}\sum_{\alpha=1}^{n} \Lambda_{nr+\alpha-1} u^{nr+\alpha-1}.
\end{aligned} \tag{4.16}$$

The commutator of $\Lambda_{nr+\alpha-1}$'s can be computed as $[\Lambda_{nr+\alpha-1}, \Lambda_{nr+\beta-1}] = (nr + \alpha - 1)\delta_{r+s,0}\delta_{\alpha+\beta,0} \cdot 1$. By the *analytic continuation*, u or z can be extended from $|u| = 1$ $(\theta \in \mathbb{R})$ or $|z| = 1$ $(\phi \in \mathbb{R})$ to the complex plane u $(\theta \in \mathbb{C})$ or z $(\phi \in \mathbb{C})$.

We rotate a circle on a complex plane u and z, respectively. Then with the use of the relations $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta} = u\frac{\mathrm{d}}{\mathrm{d}u}$ and $-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\phi} = z\frac{\mathrm{d}}{\mathrm{d}z}$ we obtain

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}u^{nr+\alpha-1} = (nr + \alpha - 1)u^{nr+\alpha-1} \qquad -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\phi}u^{nr+\alpha-1} = \left(r + \frac{\alpha-1}{n}\right)u^{nr+\alpha-1}. \tag{4.17}$$

From the TDHF viewpoint, if we assume $\phi = -\omega_c t$ on $\mathbb{C}[z, z^{-1}]$, we have

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}u^{nr+\alpha-1} = -\dot{\phi}\left(r + \frac{\alpha-1}{n}\right)u^{nr+\alpha-1} = \omega_c\left(r + \frac{\alpha-1}{n}\right)u^{nr+\alpha-1}. \tag{4.18}$$

Thus each of the infinite-dimensional fermions proves that of the fermion-harmonic oscillators with *degree* (*energy*) $(nr+\alpha-1)$ owing to the rotation of the gauge. Each shift operator plays a part of bosons carrying *energy* $(nr+\alpha-1)$ among the fermion-harmonic oscillators. The set of x_k (4.2) means coordinates of boson-harmonic oscillators which become the coordinate system on the τ-functional space. Through a scaling parameter ρ $(\in \mathbb{C}\backslash 0)$ we give the correspondence of the one-component fermion operator $(\tilde{c}(p^{-1}), \tilde{c}^\dagger(p^{-1}))$ to the generating series (4.4) in the sense of analytical continuation,

$$\begin{aligned} &p = \rho u^{-1} \stackrel{d}{=} p(\theta = 0)u^{-1} = p(\theta = 0)\,\mathrm{e}^{-\mathrm{i}\theta} \\ &(\tilde{c}(\rho^{-1}u), \tilde{c}^\dagger(\rho^{-1}u)) = (\tilde{c}(p^{-1}), \tilde{c}^\dagger(p^{-1})) \mapsto (\Psi^*(p), \Psi(p)). \end{aligned} \tag{4.19}$$

For various discrete values of p we take $p_i = p_i(\theta = 0)u^{-1}$ $(i = 1, 2, \ldots)$ with a common gauge factor u.

From the TDHF theoretical viewpoint, on the space $\mathbb{C}[z, z^{-1}]$ except Λ_0, the shift carried by the shift operator $\Lambda_{\pm(nr+\alpha-1)}$ is classified into $\pm\frac{\alpha-1}{n}$ with $r = 0$ called the intrinsic shift (Fermi), $\pm|r|$ with $\alpha - 1 = 0$ called the Laurent shift (Bose) and $\pm(|r| + \frac{\alpha-1}{n})$ otherwise, the coupling shift (Fermi $\oplus$ Bose). The Laurent shift is just a ladder shift by collective bosons. If a label of Λ on $\mathbb{C}[u, u^{-1}]$ takes k without period n, we cannot classify it as the above but have the KP hierarchy. If we impose conditions of the reduction to sl_n

(i) $\quad [X_a, \Lambda_{nr}] = 0$

and

(ii) $\quad \sum_{\alpha=1}^{n} a_{nr+\alpha-1,n(r+s)+\alpha-1} = 0 \qquad (s \in \mathbb{Z})$

then we have only to return to the original space $\mathbb{C}[z, z^{-1}]$. The Λ_{nr} has a conserved quantity on the group orbit $U(\hat{g} = \mathrm{e}^{X_a})|m\rangle$ $(X_a \in \widehat{sl_n})$ due to (i). No dependence of the τ-function on x_{nr} $(r > 0)$ appears. In the TDHF theory we use the Λ_{nr} $(r \neq 0)$ like up and down operators of collective excitations. Owing to the scaling (4.19) and the generating series (4.4), the last line in (4.16) is divided into the three classes

$$\begin{aligned} :\Psi(p)\Psi^*(p): = :\tilde{c}^\dagger(p^{-1})\tilde{c}(p^{-1}): &= \sum_{\alpha=1}^{n} \Lambda_{\pm(\alpha-1)} p^{\mp(\alpha-1)}(\theta = 0)u^{\pm(\alpha-1)} \\ &+ \sum_{r>0} \Lambda_{\pm nr} p^{\mp nr}(\theta = 0)u^{\pm nr} \\ &+ \sum_{r>0} \sum_{\alpha-1\neq 0} \Lambda_{\pm(nr+\alpha-1)} p^{\mp(nr+\alpha-1)}(\theta = 0)u^{\pm(nr+\alpha-1)}. \end{aligned} \tag{4.20}$$

In the soliton theory $p^n(\theta = 0)$ $(r = 1)$ means a spectral parameter of the iso-spectral equation and $p^{nr}(\theta = 0)$ $(r > 1)$ restricts a differential equation governing the Baker function ϕ_W into an equation $\frac{\partial}{\partial x_{nr}}\phi_W = p^{nr}(\theta = 0)\phi_W$ [22]. In the TDHF theory the exponent r of z^{-r} $(= u^{-nr})$ $(r > 0)$ means the number of excited bosons. Then we can see the close connection between the collective variables η and $\eta^*(\phi = 0)$ and the spectral parameter $p^n(\theta = 0)$. The fully parametrized SCF Hamiltonian $H_{\mathrm{F}_\infty:\,\mathrm{HF}}$ has a value on $\widehat{u_n}$ but not on $\widehat{su_n}$. Therefore, we have to remove components not compatible with condition (ii) contained in the $H_{\mathrm{F}_\infty:\,\mathrm{HF}}$ by assigning them to $\sum_{s\in\mathbb{Z}} : \psi_{ns+\alpha}\psi^*_{ns+\alpha} :$ and the conserved quantity Λ_{nr}. We call these conditions *compatible conditions for the particle and collective modes*. Then we obtain the concept of the particle and collective motion by taking a value of z $(= u^n)$ as $z = \mathrm{e}^{\mathrm{i}\phi(t)}$ and by adopting a *special choice* $\phi(t) = -\omega_c t$. It gives the collective motion as a motion of the gauge of the fermions.

The above discussions bring about the following: evolution of the time variable t yields the trajectory of the SCF Hamiltonian $H^p_{F_\infty}(x, \widetilde{\partial}_x, \hat{g})$. Then particle motion appears as time evolution of the τ-function which corresponds to that of the parameters of solutions in the soliton equations and the collective one with only one normal mode as oscillation of the τ-function through the common gauge factor z $(= u^n)$. It should be noted that this oscillation can be observed only through conserved quantities Λ_{nr}.

Standing on the above observation, further research must be made to obtain the collective submanifold selected by the SCF Hamiltonian and also to obtain a more explicit relation between the collective variable and spectral parameter. We will discuss them elsewhere with the use of a simple model, for example, the famous Lipkin model.

5. Summary and concluding remarks

Subgroup orbits made up of a *loop* path exist infinitely in Gr_m and the τ-functional method is recognized as an algebraic tool to classify them in Gr_m. To go beyond the perturbative method with respect to collective variables, we have constructed the SCF (TDHF) theory on the associative affine Kac–Moody algebra along the soliton theory using infinite-dimensional fermions. Their operators have been introduced through a Laurent expansion of finite-dimensional fermion operators with respect to degrees of freedom of the fermions related to the mean-field potential. A finite-dimensional Grassmannian Gr_m is identified with an infinite one which is affiliated with the manifold obtained by reduction to su_n of gl_∞ (reduced KP hierarchy). In this sense an algebraic treatment of extracting subgroup orbits with z $(|z| = 1)$ from the Gr_m exactly forms the differential equation (Hirota's bilinear equation) for su_n $(\subset sl_n)$ reduced KP hierarchy. The SCF theory on the F_∞ results in a *gauge theory of fermions* and *collective motion* due to quantal fluctuations of the self-consistent mean-field potential is attributed to *motion of the gauge of fermions* in which the *common gauge factor* causes interference among fermions. *A concept of particle and collective motions* is regarded as the *compatible condition for particle and collective modes*. The collective variables may have a close relation with a spectral parameter in soliton theory. These show that the SCF theory in τ-functional space F_∞ presents us with a *new algebraic method on* S^1 for microscopic understanding of fermion many-body systems.

Though state functions dependent on S^1 have a crucial role for construction of the infinite-dimensional fermions, an assumption on time-periodic collective motion is not necessarily important. Prescribing the fermions to form pairs by absorbing a *change of gauges*, the SCF Hamiltonian made up of only $H_{F_\infty;\mathrm{HF}}$ is induced and non-dispersive behaviour of a path on Gr_m can be described. Through the compatible condition for particle and collective modes, the special choice of z makes the fermion gauge periodic. We have some expressions for the pair operators of infinite-dimensional fermions in terms of Laurent spectra. This shows the close connection between the expressions for the infinite-dimensional fermions by the present theory and the finite ones by the $SO(2n)$ and $SO(2n+1)$ theories [23–26]. The above prescription gives an explanation of questions of why the fermions prefer such pairs and why infinite-dimensional Lie algebras work well in fermion many-body systems. Recently, another infinite-dimensional algebraic approach related intimately to ours has been developed and the exact solution for the pairing Hamiltonian obtained [27].

It must be stressed that the TDHF theory on F_∞ describes dynamics on real fermion-harmonic oscillators but that the soliton theory does so on complex fermion-harmonic oscillators. This remark suggests that it is an important task to extend the TDHF theory on real space $\widehat{su_n}$ to that on complex space $\widehat{sl_n}$. It will demand a deeper understanding of the concept of quasi-particle energy and the boson one, namely the independent-particle and mean-field potential, standing on an algebro-geometric viewpoint. This also proposes the problem of the connection between this paper and the resonating mean-field theories [28]. It is very interesting to study new motion on a complex Grassmannian in finite fermion many-body systems. We have assumed here only one circle. The TDHF equation on $\tau_m(x, \hat{g})$, however, should lead to multi-circles. It relates closely to problems of multi-dimensional soliton theory.

We have made clear a unified aspect between the SCF method and τ-functional method through the abstract fermion Fock space on S^1. It means that algebro-geometric structures of *infinite*-dimensional fermion many-body systems are also realizable in *finite*-dimensional ones. Finally, we point out that to improve drawbacks in the perturbative SCF method it is useful to find a way to construct infinite-dimensional boson variables x_k from original group-parameters of $g \in U(n)$ [29].

To both the questions suggested by Tajiri in his acknowledgements, we cannot give a satisfactory answer yet within the present framework, because both the two methods mentioned below are *a priori* based on the fermion system from the outset. That is to say, the SCF method describes a quasi-classical dynamics on the Grassmannian (the Slater determinantal orbit) which is induced owing to the anticommutative property of fermions. On the other hand, the τ-functional method also uses the fermions to explain the Grassmannian of solution space which reflects the fermion-like behaviour of soliton solutions. Therefore, we should study further why extraction of soliton equations out of classical wave equations brings out the Grassmannian.

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